

SCATTERING THEORY AND THE GEOMETRY OF MULTI-TWISTOR SPACES

BY

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ABSTRACT. Existing results which show the zero rest mass field equations to be encoded in the geometry of projective twistor space are extended, and it is shown that the geometries of spaces of more than one twistor contain information concerning the scattering of such fields. Some general constructions which describe spacetime interactions in terms of cohomology groups on subvarieties in twistor space are obtained and are used to construct a purely twistorial description of spacetime propagators and of first order ϕ^4 scattering. Spacetime expressions concerning these processes are derived from their twistor counterparts, and a physical interpretation is given for the twistor constructions.

1. Introduction. Twistor space was introduced by Roger Penrose in 1967 [18] as a new arena in which to analyze the behavior of conformally invariant systems.

Over the past fourteen years (see, for example, Penrose [21 or 24]), this program has met with a great deal of success, and it has been possible to describe a variety of physical phenomena in terms of twistor geometry. Penrose's nonlinear graviton construction [22], for example, describes self-dual solutions of the vacuum Einstein equations in terms of deformations of twistor space. Ward [31] has described self-dual Yang-Mills fields in terms of vector bundles over twistor space, and this result has led to the solution of the Yang-Mills equations on S^4 by Atiyah, Hitchin, Drinfeld and Manin [1].

Penrose also shows in [19 and 20] that the zero rest mass field equations are encoded in the geometry of projective twistor space. A considerable refinement of this work appears in Eastwood, Penrose and Wells [6]. It has been suggested by Penrose [25] that these ideas may be extended to deal with interactions of massless fields, and such an extension is the aim of this paper.

§2 will give a brief description of twistor space and summarize the work in [6]. The inner product pairing will be described in §3, and it will be shown that this pairing can be described in terms of the geometrical structure of the product of two twistor spaces. This work, from an analytic rather than a geometric point of view, also appears in [5].

The main result of the paper is Theorem 4.1, which generalizes the inner product construction to one which can be applied to arbitrary products of projective twistor spaces. As an example, a geometrical description of first order ϕ^4 scattering is given.

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§5 reinterprets the earlier constructions in terms of nonprojective twistor spaces and uses the results to define a “universal” propagator which propagates fields of any positive helicity.

§6 returns to spacetime, giving explicit calculations of the free space spin-0 propagators for massless fields and of the ϕ^4 amplitudes. Symmetries of the free space helicity $n/2$ propagators are discussed, and we conclude by giving a physical interpretation of the geometrical objects with which we have been dealing in §7.

The results of this paper are of two types. *Theorems* deal with tools and constructions which are purely mathematical in nature, while *propositions* discuss their applications to physical situations. This paper is an extension of earlier work in [5, 8, 9 and 10]. The author would like to thank Mike Eastwood, Andrew Hodges, Stephen Huggett and Roger Penrose for many illuminating discussions.

2. The Penrose transform. *Twistor space* $\mathbf{T}$ is a 4-dimensional complex vector space equipped with a Hermitian form Φ of signature $(+, +, -, -)$. *Projective twistor space* $\mathbf{P} = \mathbf{P}(\mathbf{T})$ is the space of lines in $\mathbf{T}$, and is thus isomorphic to complex projective 3-space $\mathbf{CP}^3$. *Complexified compactified Minkowski space* $\mathbf{M}$ is the Grassmannian of 2-planes (i.e., 2-dimensional complex vector spaces) in $\mathbf{T}$. If we define the *projective primed spin bundle* $\mathbf{F}$ to be the flag manifold of lines in 2-planes in $\mathbf{T}$, we get natural projections:

$$\begin{array}{ccc} & \mathbf{F} & \\ \mu \swarrow & & \searrow \nu \\ \mathbf{P} & & \mathbf{M} \end{array}$$

A point $x \in \mathbf{M}$ gives rise to a line $L_x \equiv \mu\nu^{-1}(x)$ in $\mathbf{P}$, and it is not hard to see that all lines in $\mathbf{P}$ are of this form. Two points $x, y \in \mathbf{M}$ are null-separated if and only if the associated lines L_x and L_y intersect, and it is this observation which underlies the analysis of conformally invariant systems using twistor geometry.

We will denote a twistor by $Z^\alpha = (Z^0, Z^1, Z^2, Z^3)$; taking the Z^α to be homogeneous coordinates on $\mathbf{CP}^3$; we will use Z^α to denote a projective twistor as well. It will often be useful to rewrite a twistor (or projective twistor) as a pair of spinors [17], $Z^\alpha = (\omega^A, \pi_{A'})$. Points of complexified Minkowski space $\mathbf{M}^I$ will be denoted by $x^a = (x^0, x^1, x^2, x^3)$, or by $x^{AA'}$ as in [17].

For U an arbitrary region in $\mathbf{M}$, we will denote $\mu\nu^{-1}(U)$ by U'' , following [6]. For example, if U is the forward tube

$$\mathbf{M}^+ = \{x^a - iy^a \in \mathbf{M}^I \text{ such that } x^a, y^a \text{ are real and } y^a \text{ is timelike and future pointing}\},$$

it can be shown that $\mathbf{M}^{+''} = \mathbf{P}^+$, the set of projective twistors satisfying $\Phi(Z^\alpha) > 0$. Similarly, for the backward tube

$$\mathbf{M}^- = \{x^a - iy^a \in \mathbf{M}^I \text{ such that } x^a, y^a \text{ are real and } y^a \text{ is timelike and past pointing}\},$$

we have $\mathbf{M}^{-''} = \mathbf{P}^-$, the set of projective twistors such that $\Phi(Z^\alpha) < 0$.

We denote the dual of $\mathbf{T}$ (as a complex vector space) by $\mathbf{T}^*$, and set $\mathbf{P}^* \equiv \mathbf{P}(\mathbf{T}^*)$; an element of dual twistor space will be written $W_\alpha = (\eta_A, \xi^{A'})$. Lines in $\mathbf{P}^*$ again

correspond to points in $\mathbf{M}$; for a given $x \in \mathbf{M}$, the associated line in $\mathbf{P}^*$ will be denoted $L_x^\perp$, since it is easy to see that $L_x^\perp$ is the line in $\mathbf{P}^*$ orthogonal to L_x . For U a region in $\mathbf{M}$, the associated region in $\mathbf{P}^*$ will be denoted ${}''U$; ${}''\mathbf{M}^+ = \mathbf{P}^{*-}$, and ${}''\mathbf{M}^- = \mathbf{P}^{*+}$.

All of these ideas are explained in more depth and with greater clarity in Wells [32] or in Hughston and Ward [15]. The latter reference especially contains information on a wide variety of topics in twistor theory.

Let $U \subset \mathbf{M}^I$. By a *zero rest mass (zrm) field* on U of helicity $n/2$, we will mean, as in [17], a spinor-valued symmetric holomorphic function ϕ on U satisfying:

$$(2.1) \quad \begin{aligned} \nabla^{AA'} \phi_{A' \dots B'} &= 0 \quad \text{for } n > 0, \\ \square \phi &= 0 \quad \text{for } n = 0, \\ \nabla^{AA'} \phi_{A' \dots B} &= 0 \quad \text{for } n < 0 \end{aligned}$$

where ϕ has $|n|$ spinor indices, $\nabla^{AA'} = \partial/\partial x_{AA'}$, and $\square = \nabla^{AA'} \nabla_{AA'}$. We will denote by $Z_n(U)$ the group of helicity $n/2$ zrm fields on U .

We now turn to projective twistor space, and let ξ denote the hyperplane section bundle over $\mathbf{P}$. If we fix dual twistors A_α and B_α , transition functions for $\xi \in H^1(\mathbf{P}^-; \mathcal{O}^*)$ are given by

$$(2.2) \quad f_{12} = \frac{Z \cdot A}{Z \cdot B},$$

where $Z \cdot A$ denotes $Z^\alpha A_\alpha$, etc., and f_{12} is defined on the intersection of the sets $U_1 = \{Z \cdot A \neq 0\}$ and $U_2 = \{Z \cdot B \neq 0\}$, which cover $\mathbf{P}^-$ if the line joining A and B in $\mathbf{P}^*$ lies entirely in $\mathbf{P}^{*-}$. We can represent ξ as an element of $H^1(\mathbf{P}; \mathcal{O}^*)$ similarly, by using more than two sets.

We write $\mathcal{O}(k)$ for the sheaf over $\mathbf{P}$ of germs of sections of ξ^{-k} . A section f of $\mathcal{O}(k)$ can be thought of as a holomorphic function of a nonprojective twistor, homogeneous of degree k in its argument:

$$(2.3) \quad Z^\alpha \partial f / \partial Z^\alpha = k f$$

($Z^\alpha \partial / \partial Z^\alpha$ is the homogeneity operator on twistor space). Similar sheaves are defined on $\mathbf{P}^*$.

It was discovered by Penrose [23] that zrm fields can be naturally described in terms of elements of sheaf cohomology groups on $\mathbf{P}$, and a comprehensive discussion of this matter can be found in [6]. The main result of that paper is as follows: Let $U \subset \mathbf{M}$. U will be called *suitable* if:

- (1) U is Stein;
- (2) $H^1(U; \mathbf{Z}) = H^2(U; \mathbf{Z}) = 0$; and
- (3) for any $Z^\alpha \in U''$, $\nu\mu^{-1}(Z^\alpha) \cap U$ is connected and $H^1(\nu\mu^{-1}(Z^\alpha) \cap U; \mathbf{Z}) = 0$; and similarly for any $W_\alpha \in {}''U$.

PROPOSITION 2.1. *Suppose that $U \subset \mathbf{M}$ is suitable. Then there are natural isomorphisms*

$$(2.4) \quad H^1(U''; \mathcal{O}(-n-2)) \simeq Z_n(U) \simeq H^1({}''U; \mathcal{O}(n-2)).$$

PROOF. See [6]. $\square$

The isomorphisms of Proposition 2.1 are known collectively as the *Penrose transform*. Since $\mathbf{M}^+$ and $\mathbf{M}^-$ are easily seen to be suitable, we have

COROLLARY 2.2.

$$(2.5) \quad \begin{aligned} H^1(\mathbf{P}^+; \mathcal{O}(-n-2)) &\simeq Z_n(\mathbf{M}^+) \simeq H^1(\mathbf{P}^{*-}; \mathcal{O}(n-2)), \\ H^1(\mathbf{P}^-; \mathcal{O}(-n-2)) &\simeq Z_n(\mathbf{M}^-) \simeq H^1(\mathbf{P}^{*+}; \mathcal{O}(n-2)). \end{aligned} \quad \square$$

The description in [6] shows clearly that the zrm field equations (2.1) have been encoded into the geometry of the mapping $\mu: \mathbf{F} \rightarrow \mathbf{P}$.

Later in this paper, we will also need a description of massless fields in terms of cohomology groups on regions of *nonprojective* twistor space. This problem has been addressed by Eastwood [4], and the remaining results of this section are his.

Let $\pi: \mathbf{T} - \{0\} \rightarrow \mathbf{P}$ be the natural projection. For a region $U \subset \mathbf{P}$, we will denote $\pi^{-1}(U)$ by $\tilde{U}$. Motivated by (2.3), we define “homogeneous” sheaves $\mathcal{H}(n)$ via the short exact sequence on $\mathbf{T}$:

$$(2.6) \quad \begin{aligned} 0 \rightarrow \mathcal{H}(n) \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0, \\ f \rightarrow Z \cdot \partial f / \partial Z - nf \end{aligned}$$

so that sections of $\mathcal{H}(n)$ are homogeneous functions of the twistor Z^α .

THEOREM 2.3 (EASTWOOD). *For $U \subset \mathbf{P}$, there is an exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(U; \mathcal{O}(n)) \rightarrow H^1(\tilde{U}; \mathcal{H}(n)) \rightarrow \Gamma(U; \mathcal{O}(n)) \rightarrow H^2(U; \mathcal{O}(n)) \\ \rightarrow H^2(\tilde{U}; \mathcal{O}(n)) \rightarrow H^1(U; \mathcal{O}(n)) \rightarrow \dots \end{aligned}$$

PROOF. The direct images of the sheaves $\mathcal{H}(n)$ can be evaluated by Laurent expanding an arbitrary holomorphic function along the fibers of π . Application of the Leray spectral sequence [11, §II.4.17] and the generalized Gysin cohomology sequence [11, §I.4.6 or 28, §9.5] then gives the desired result. Details are in [4]. $\square$

3. Twistor propagators. Suppose we are given two massless fields

$$(3.1) \quad \psi \in Z_n(\overline{\mathbf{M}}^+) \quad \text{and} \quad \theta \in Z_n(\overline{\mathbf{M}}^-).$$

There is then a well-known pairing [7] which assigns to ψ and θ a complex number known as their *inner product* which we will denote $\langle \theta | \psi \rangle$. (Our notation here is nonstandard, as $\langle \theta | \psi \rangle$ is linear in both of its arguments. It is more usual for it to be complex conjugate linear in the first argument, so that the inner product of the states in (3.1) would be $\langle \bar{\theta} | \psi \rangle$, with $\bar{\theta} \in Z_n(\overline{\mathbf{M}}^+)$.) By Corollary 2.2 (more accurately, its analog on the closures of the forward and backward tubes), we conclude the following.

PROPOSITION 3.1. *For any integer n , there is a natural pairing*

$$(3.2) \quad H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2)) \otimes H^1(\overline{\mathbf{P}}^-; \mathcal{O}(n-2)) \rightarrow \mathbf{C}. \quad \square$$

It would be more satisfactory to have a purely twistorial proof of this result, rather than one depending on a spacetime evaluation of the inner product as in [7]. This should be possible, since if $f(Z)$ and $g(Z)$ are representative cocycles for the

cohomology elements in (3.2), the scalar product of the corresponding states is given (informally) by

$$(3.3) \quad \oint f(Z)g(Z) DZ,$$

where $DZ = \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$ is the canonical (up to choice of ε) $(3, 0)$ -form homogeneous of degree 4 on $\mathbf{CP}^3$. We will denote by ε the nonprojective version of this form; $\varepsilon = \varepsilon_{\alpha\beta\gamma\delta} dZ^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \in \Lambda^4(\mathbf{T})$.

THEOREM 3.2. *Let X be a complex manifold, and $\mathcal{S}$ and $\mathcal{T}$ sheaves over X . Then for U, V open in X , there is a natural pairing*

$$(3.4) \quad H^p(U; \mathcal{S}) \otimes H^q(V; \mathcal{T}) \rightarrow H^{p+q+1}(U \cup V; \mathcal{S} \otimes \mathcal{T}),$$

denoted

$$(f, g) \rightarrow f \cdot g.$$

This pairing has the following properties:

- (1) *If f is the restriction to U of an element of $H^p(U \cup V; \mathcal{S})$, then $f \cdot g = 0$.*
- (2) *$f \cdot g = (-1)^{p+1} g \cdot f$.*

PROOF. We define $f \cdot g$ to be $\partial^*(f \cup g)$, where $f \cup g$ is the usual cup product, given by a map

$$H^p(U; \mathcal{S}) \otimes H^q(V; \mathcal{T}) \rightarrow H^{p+q}(U \cap V; \mathcal{S} \otimes \mathcal{T}),$$

and ∂^* is the coboundary operator in the Mayer-Vietoris sequence

$$(3.5) \quad \cdots \rightarrow H^k(U; \mathcal{R}) \oplus H^k(V; \mathcal{R}) \xrightarrow{\rho} H^k(U \cap V; \mathcal{R}) \xrightarrow{\partial^*} H^{k+1}(U \cap V; \mathcal{R}) \\ \xrightarrow{\rho} H^{k+1}(U; \mathcal{R}) \oplus H^{k+1}(V; \mathcal{R}) \rightarrow \cdots$$

(where $\mathcal{R} = \mathcal{S} \otimes \mathcal{T}$). Statement (1) follows from the exactness of the sequence (3.5), while (2) follows from the facts that $f \cup g = (-1)^{p+q} g \cup f$ and ∂^* is antisymmetric. $\square$

We will refer to this pairing as the *dot product*. It can be shown [8] that this operation is induced by multiplication of Čech representatives.

An obvious extension of the dot product to the closed sets $\overline{\mathbf{P}^+}$ and $\overline{\mathbf{P}^-}$ gives a precise meaning to the expression $f(Z)g(Z)$ in (3.3). To interpret the intergration, we need

THEOREM 3.3 (SERRE [26]). *Suppose that X is an m -dimensional compact complex manifold, and let κ be the canonical bundle over X . Then for any line bundle $\xi \in H^1(X; \mathcal{O}^*)$, there is a canonical isomorphism*

$$(3.6) \quad \text{Serre: } H^p(X; \mathcal{O}(\xi)) \simeq H^{m-p*}(X; \mathcal{O}(\kappa\xi^{-1})),$$

where $H^{m-p*}(X; \mathcal{S})$ denotes the dual of $H^{m-p}(X; \mathcal{S})$ as a complex vector space. $\square$

The canonical bundle over $\mathbf{CP}^m$ is Ω^m , the bundle whose sections are $(m, 0)$ -forms. The choice of (for example) the 4-twistor $\varepsilon_{\alpha\beta\gamma\delta}$ is equivalent to an identification of

Ω^3 with $\mathcal{O}(-4)$ on $\mathbf{P}$, and we have

COROLLARY 3.4. *Subject to an identification of Ω^m with $\mathcal{O}(-m-1)$ on $\mathbf{CP}^m$, there are canonical isomorphisms*

$$(3.7) \quad \text{Serre: } H^m(\mathbf{CP}^m; \mathcal{O}(-n-m-1)) \simeq H^{0*}(\mathbf{CP}^m; \mathcal{O}(n)),$$

the latter space being an $\binom{m+n}{m}$ -dimensional complex vector space. $\square$

(3.7) provides us with an isomorphism Serre: $H^3(\mathbf{P}; \mathcal{O}(-4)) \simeq \mathbf{C}$, and the inner product pairing is now given by

$$(3.8) \quad H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2)) \otimes H^1(\overline{\mathbf{P}}^-; \mathcal{O}(n-2)) \rightarrow \mathbf{C},$$

$$(f, g) \rightarrow \text{Serre}(f \cdot g).$$

In addition to (3.8), it should also be possible to interpret the inner product as a pairing

$$(3.9) \quad H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2)) \otimes H^1(\overline{\mathbf{P}}^{*-}; \mathcal{O}(-n-2)) \rightarrow \mathbf{C}.$$

In fact, this is a more natural construction than the previous one if we are interested in evaluating expressions such as $\langle \bar{\psi} | \psi \rangle$. The reason for this is that the natural interpretation on twistor space of the complex conjugation of zrm fields is as a map

$$(3.10) \quad H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2)) \rightarrow H^1(\overline{\mathbf{P}}^{*-}; \mathcal{O}(-n-2)),$$

rather than as a map

$$H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2)) \rightarrow H^1(\overline{\mathbf{P}}^-; \mathcal{O}(n-2)).$$

We will construct the pairing (3.9) by reducing this problem, in a sense, to the previous one. There are two-point fields $\phi_{-n}(x, y)$ such that for ψ and θ as in (3.1),

$$(3.11) \quad \langle \theta(x) | \psi(x) \rangle = \langle \theta(x) | \phi_{-n}(x, y) | \psi(y) \rangle.$$

In other words, the inner product of ψ and θ can be calculated by evaluating the inner product of $\psi(y)$ and $\phi_{-n}(x, y)$ to obtain a zrm field $\tilde{\psi}(x)$, and then taking the inner product of this field with θ . We will refer to the ϕ_{-n} (and to their twistor counterparts) as *propagators*, saying that they *propagate* or *mediate* the inner product.

On spacetime ϕ_0 is given by

$$(3.12) \quad \phi_0(x, y) = 1/(x-y)^2,$$

while the other ϕ_n are

$$(3.13) \quad \phi_n(x, y) = i^n \underbrace{\nabla_{AA'} \cdots \nabla_{BB'}}_{n \text{ times}} \phi_0 \quad \text{for } n \geq 0,$$

$$\phi_n(x, y) = (-i)^{-n} \underbrace{\nabla_{AA'} \cdots \nabla_{BB'}}_{-n \text{ times}} \phi_0 \quad \text{for } n \leq 0,$$

where all of the derivatives are with respect to x and the resulting fields can easily be shown to be symmetric in their spinor indices and to satisfy the zrm field equations in both x and y . Since it is clear that the ϕ_n are well behaved for $x \in \mathbf{M}^-$ and $y \in \mathbf{M}^+$ (implying $(x-y)^2 \neq 0$), we conclude that they correspond to elements

$$(3.14) \quad \phi_n \in H^2(\mathbf{P}_x^- \times \mathbf{P}_y^{*-}; \mathcal{O}(-n-2, -n-2)),$$

where we have abused notation by using the same symbol for both the spacetime and the twistor versions of the propagators. Where confusion is possible, we will write $\phi_n(x^a, y^a)$ or $\phi_n(X^\alpha, Y_\alpha)$. To obtain (3.14) we used an easy generalization of Corollary 2.2 to spaces of more than one twistor; the subscripts on the twistor spaces serve to indicate which twistor space is associated to which spacetime variable.

We now obtain (3.9) by noting that, for $f \in H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-n-2))$ and $g \in H^1(\overline{\mathbf{P}}^{*+}; \mathcal{O}(-n-2))$, $f \cdot \phi_{-n} \cdot g \in H^6(\mathbf{P} \times \mathbf{P}^*; \mathcal{O}(-4, -4)) \simeq \mathbb{C}$.

As before, this description is slightly unsatisfactory, and we would prefer to have one in which spacetime arguments do not appear. It is clearly sufficient to construct the propagators (3.14) from twistor considerations alone. As a preliminary step in this direction, we have

PROPOSITION 3.5 (EASTWOOD AND GINSBERG [5]). *The twistor propagators $\phi_n(Z^\alpha, W_\alpha)$ satisfy*

$$(3.15) \quad \frac{\partial \phi_n}{\partial Z^\alpha} = W_\alpha \phi_{n+1} \quad \text{and} \quad \frac{\partial \phi_n}{\partial W_\alpha} = Z^\alpha \phi_{n+1}.$$

PROOF. This is a straightforward matter of interpreting the relations (3.15) on spacetime. For example, the operator $\partial/\partial Z^\alpha$ gives rise to a map

$$H^1(\mathbf{P}^-; \mathcal{O}(-n-2)) \rightarrow H^1(\mathbf{P}^-; \mathcal{O}_\alpha(-n-3)),$$

and the effect of this map on the associated zrm fields is described in Penrose [21] or Eastwood [2]. Using their results, it can be shown that (3.15) is nothing more than a reformulation of (3.13); the details are in [8]. $\square$

We now have

THEOREM 3.6 (EASTWOOD AND GINSBERG [5]). *The relations (3.15) characterize the twistor propagators ϕ_n up to scale.*

PROOF. Although this is an important result, the details of the proof are of no special interest. They can be found in [5]. $\square$

We see from this theorem that the relations (3.15) lead to a purely twistorial characterization of the inner product pairing (3.9). Unfortunately this result, as it stands, is completely useless for practical calculation. In order to actually *evaluate* the inner product of two zrm fields, we need to construct the twistor propagators explicitly.

To do this, note that (3.15) implies

$$(3.16) \quad W_\alpha \frac{\partial \phi_{-2}}{\partial W_\alpha} = (Z \cdot W) \phi_{-1} = 0,$$

since ϕ_{-2} is homogeneous of degree 0 in W_α . This is in some sense to be expected, since the pairing (3.9) is given informally for $n = 1$ by (in analogy with (3.3)) [25]

$$(3.17) \quad (f, g) \rightarrow \oint \frac{f(Z)g(W)}{Z \cdot W} DZ \wedge DW,$$

and if we multiply the integrand in this expression by $Z \cdot W$, the singularity in $Z \cdot W$ vanishes and the contour over which the integral is taken becomes homologous to zero [30]. This is reflected in (3.16).

We now define *ambitwistor space* Ω [3] to be

$$(3.18) \quad \Omega \equiv \mathbf{P} \times \mathbf{P}^* \cap \{Z \cdot W = 0\},$$

and take Ω^- to be the intersection of Ω with $\mathbf{P}^- \times \mathbf{P}^{*-}$. There is a short exact sequence on $\mathbf{P}^- \times \mathbf{P}^{*-}$

$$(3.19) \quad 0 \rightarrow \mathcal{O}(m, n) \xrightarrow{Z \cdot W} \mathcal{O}(m+1, n+1) \xrightarrow{\rho} \mathcal{O}_{\Omega^-}(m+1, n+1) \rightarrow 0,$$

leading to a long exact sequence containing the segment

$$(3.20) \quad H^p(\Omega^-; \mathcal{O}(m+1, n+1)) \xrightarrow{\delta} H^{p+1}(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathcal{O}(m, n)) \\ \xrightarrow{Z \cdot W} H^{p+1}(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathcal{O}(m+1, n+1)).$$

We now have the following.

LEMMA 3.7. *Suppose $\psi \in H^p(\Omega^-; \mathcal{O}(m+1, n+1))$ extends to an open neighborhood of Ω^- in $\mathbf{P}^- \times \mathbf{P}^{*-}$. Then*

$$(3.21) \quad \delta\psi = \frac{1}{Z \cdot W} \cdot \psi.$$

PROOF. This is an easy consequence of the Čech description of the dot product which was given in the paragraph following the proof of Theorem 3.2. $\square$

We also have

PROPOSITION 3.8. *There exists a $\phi \in H^1(\Omega^-; \mathcal{O})$ such that $\phi_{-1} = \delta\phi$.*

PROOF. This is a consequence of (3.16) and the exactness of (3.20). $\square$

These two results are very suggestive. If ϕ in Proposition 3.8 were to extend to an open neighborhood of Ω^- in $\mathbf{P}^- \times \mathbf{P}^{*-}$, we could then form, for $n \geq -1$,

$$(3.22) \quad \phi_n = \frac{(-1)^{n+1}(n+1)!}{(Z \cdot W)^{n+2}} \cdot \phi,$$

and would have, for example,

$$\frac{\partial \phi_n}{\partial Z^\alpha} = \frac{(-1)^n(n+2)!}{(Z \cdot W)^{n+3}} W_\alpha \cdot \phi + \frac{(-1)^{n+1}(n+1)!}{(Z \cdot W)^{n+2}} \frac{\partial \phi}{\partial Z^\alpha} = W_\alpha \phi_{n+1},$$

provided that $\partial\phi/\partial Z^\alpha = 0$. This leads to the following.

PROPOSITION 3.9. *Suppose that $\phi \in H^1(U; \mathcal{O})$ for some neighborhood U of Ω^- in $\mathbf{P}^- \times \mathbf{P}^{*-}$ satisfies*

$$(3.23) \quad \frac{\partial \phi}{\partial Z^\alpha} = 0 = \frac{\partial \phi}{\partial W_\alpha}.$$

The twistor propagators ϕ_n are then given, up to scale, by (3.22) for $n \geq -1$. $\square$

The relations (3.23) suggest that ϕ is an invariant which can be obtained by examining the geometry of the space Ω^- . We think of Ω^- as a fiber bundle, with base $\mathbf{P}^{*-}$ and fiber the intersection of $\mathbf{P}^-$ with the plane in $\mathbf{P}$ dual to a fixed

$W_\alpha \in \mathbf{P}^{*-}$. Since such a plane contains a line in $\mathbf{P}^+$ but not one in $\mathbf{P}^-$, it is not hard to see that the fibers are contractible, so that $H^p(\Omega^-; \mathbf{Z}) \simeq H^p(\mathbf{P}^{*-}; \mathbf{Z})$.

$\mathbf{P}^*$ is itself a fiber bundle, with base S^4 and fiber S^2 . $\mathbf{P}^{*-}$ is that portion of this fibration lying over a hemisphere E^4 , and it follows that $H^p(\mathbf{P}^{*-}; \mathbf{Z}) \simeq H^p(S^2; \mathbf{Z})$. Thus $H^0(\Omega^-; \mathbf{Z}) \simeq \mathbf{Z} \simeq H^2(\Omega^-; \mathbf{Z})$, and the remaining groups vanish. Similar considerations show that $H^2(\Omega; \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}$, a result which will be of use to us later.

The short exact sequence on Ω^-

$$(3.24) \quad 0 \rightarrow \mathbf{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \rightarrow 0,$$

$$f \rightarrow \exp(2\pi i f)$$

therefore gives rise to an exact sequence

$$H^1(\Omega^-; \mathbf{Z}) = 0 \rightarrow H^1(\Omega^-; \mathcal{O}) \rightarrow H^1(\Omega^-; \mathcal{O}^*) \rightarrow \mathbf{Z} = H^2(\Omega^-; \mathbf{Z}).$$

It follows that ϕ corresponds to some line bundle $e(\phi)$ on Ω^- with vanishing Chern class.

Since $\mathcal{O}(m, n)$ has Chern class $m + n$ over Ω^- , a natural choice for this line bundle is $\mathcal{O}(1, -1)$. If we fix twistors A^α and B^α such that the line joining them lies entirely in $\mathbf{P}^+$, transition functions for $\mathcal{O}(1, -1)$ are given by

$$\frac{A \cdot W}{B \cdot W} \frac{Z \cdot \bar{B}}{Z \cdot \bar{A}},$$

and these actually define a line bundle with vanishing Chern class in a neighborhood of Ω^- . We therefore set

$$(3.25) \quad \phi = \frac{1}{2\pi i} \log \left(\frac{A \cdot W}{B \cdot W} \frac{Z \cdot \bar{B}}{Z \cdot \bar{A}} \right);$$

verification that $\partial\phi/\partial Z^\alpha$ and $\partial\phi/\partial W_\alpha$ are coboundaries is now straightforward. It follows that the ϕ_n are given explicitly by

$$(3.26) \quad \phi_n = \frac{(-1)^{n+1}(n+1)!}{2\pi i (Z \cdot W)^{n+2}} \cdot \log \left(\frac{A \cdot W}{B \cdot W} \frac{Z \cdot \bar{B}}{Z \cdot \bar{A}} \right)$$

for $n \geq -1$. (A version of this result appears in Sparling [29].) We will often write this as

$$(3.27) \quad \phi_n = (Z \cdot W)_{n+2} \cdot \log(\mathcal{O}(1, -1)),$$

where

$$(3.28) \quad (x)_k \equiv \frac{(-1)^{k+1}(k-1)!}{x^k}$$

are known as *bracket factors* [25], and the factor of $1/2\pi i$ is incorporated into the notation $\log(\mathcal{O}(1, -1))$.

4. Geometric constructions. The construction of the twistor propagators at the end of the last section is a special case of a much more general procedure. Here is the basic result.

THEOREM 4.1. *Let X be a complex manifold such that the Chern mapping*

$$c: H^{p-1}(X; \mathcal{O}^*) \rightarrow H^p(X; \mathbf{Z})$$

is surjective for a fixed p . Let ξ be a line bundle over X , and suppose that $A \subset X$ is the zero set of some section $f \in \Gamma(X; \xi)$. If $H^{p-1}(A; \mathbf{Z}) = 0$, then for any integer $n \geq 0$, there is a natural map

$$(4.1) \quad P_n: H^p(X, A; \mathbf{Z}) \rightarrow H^p(X; \mathcal{O}(\xi^{-n-1})).$$

We will denote P_0 by P .

PROOF. The cohomology sequence of the pair (X, A) is, in part,

$$H^{p-1}(A; \mathbf{Z}) = 0 \rightarrow H^p(X, A; \mathbf{Z}) \rightarrow H^p(X, \mathbf{Z}) \xrightarrow{\rho} H^p(A; \mathbf{Z}),$$

so that $H^p(X, A; \mathbf{Z}) \simeq \ker \rho: H^p(X, \mathbf{Z}) \rightarrow H^p(A; \mathbf{Z})$. Let $g \in H^p(X, A; \mathbf{Z}) \subset H^p(X; \mathbf{Z})$; we have the commutative diagram

$$\begin{array}{ccccccc} H^{p-1}(X; \mathcal{O}) & \rightarrow & H^{p-1}(X; \mathcal{O}^*) & \xrightarrow{c} & H^p(X; \mathbf{Z}) & \rightarrow & 0 \\ \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \\ H^{p-1}(A; \mathbf{Z}) = 0 & \rightarrow & H^{p-1}(A; \mathcal{O}) & \xrightarrow{e} & H^{p-1}(A; \mathcal{O}^*) & \xrightarrow{c'} & H^p(A; \mathbf{Z}) \rightarrow 0 \end{array}$$

where the rows are exact, and can therefore find a $\tilde{g} \in H^{p-1}(X; \mathcal{O}^*)$ with $c\tilde{g} = g$. Since $c'\rho\tilde{g} = \rho g = 0$, there is a unique $h \in H^{p-1}(A; \mathcal{O})$ such that $e(h) = \rho\tilde{g}$.

It is easy to see that h is well defined on a neighborhood of A in X , and we define

$$(4.2) \quad P_n(g) \equiv (f)_{n+1} \cdot h \in H^p(X; \mathcal{O}(\xi^{-n-1})).$$

Since

$$H^{p+1}(X; \mathcal{O}) \subset \ker[\cdot (f)_{n+1}]: H^{p-1}(A; \mathcal{O}) \rightarrow H^p(X; \mathcal{O}(\xi^{-n-1})),$$

$P_n(g)$ is independent of the choice of g made in defining h . $\square$

The key geometrical group in this theorem is the relative cohomology group $H^p(X, A; \mathbf{Z})$, rather than $H_p(X - A; \mathbf{Z})$, which has been more usually investigated in this sort of problem [30]. This difference is more apparent than real, since p will generally be equal to the complex dimension of the space X , in which case these groups are isomorphic to each other by Lefschetz duality [28].

As an application, we have

PROPOSITION 4.2. *Let $X = \mathbf{P}^- \times \mathbf{P}^{*-}$, and $\xi = \mathcal{O}(1, 1)$. Then the zero set of $f = Z \cdot W$ is Ω^- , and $H^2(\mathbf{P}^- \times \mathbf{P}^{*-}, \Omega^-; \mathbf{Z}) \simeq \mathbf{Z}$. If k is the generator of this group, then*

$$\phi_n = P_{n+1}(k) \in H^2(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathcal{O}(-n-2, -n-2)).$$

PROOF. $H^2(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}$, and it is not hard to show that restriction takes $(m, n) \in H^2(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathbf{Z})$ to $m + n \in H^2(\Omega^-; \mathbf{Z}) \simeq \mathbf{Z}$. $(1, -1)$ therefore generates the relative cohomology group $H^2(\mathbf{P}^- \times \mathbf{P}^{*-}, \Omega^-; \mathbf{Z})$, and since the Chern class of the line bundle $\mathcal{O}(m, n)$ over $\mathbf{P}^- \times \mathbf{P}^{*-}$ is (m, n) , we see that we can

take $\tilde{g} = \mathcal{O}(1, -1)$ in the proof of Theorem 4.1. The remainder of that proof is now seen to be a simple generalization of the construction at the end of the last section.

□

Recalling (3.27), the inner product pairing of (3.9) is now given as

$$(4.3) \quad H^1(\overline{\mathbf{P}^+}; \mathcal{O}(-n-2)) \otimes H^1(\overline{\mathbf{P}^{*-}}; \mathcal{O}(-n-2)) \rightarrow \mathbb{C},$$

$$(f, g) \rightarrow \text{Serre}(f \cdot (Z \cdot W)_{-n+2} \cdot \phi \cdot g)$$

and, in light of condition (1) of Theorem 3.2, we see that necessary conditions for this pairing to be nondegenerate are that $\phi \in H^1(\Omega^-; \mathcal{O})$ does not extend to an element of $\mathbf{P}^- \times \mathbf{P}^{*-}$ and that the ϕ_n do not extend to all of $\mathbf{P} \times \mathbf{P}^*$.

The reason $\log(\mathcal{O}(1, -1))$ does not extend to $\mathbf{P}^- \times \mathbf{P}^{*-}$ is simply that the line bundle $\mathcal{O}(1, -1)$ does not have vanishing Chern class on $\mathbf{P}^- \times \mathbf{P}^{*-}$, and therefore cannot be pulled back along the map e in (3.24). Meanwhile, if we try to use Theorem 4.1 to extend ϕ_n to all of $\mathbf{P} \times \mathbf{P}^*$, we find that since $H^2(\Omega; \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z} \simeq H^2(\mathbf{P} \times \mathbf{P}^*; \mathbf{Z})$,

$$(4.4) \quad H^2(\mathbf{P} \times \mathbf{P}^*, \Omega; \mathbf{Z}) \simeq 0,$$

and the theorem cannot be applied. A careful investigation of this problem reveals that the difficulty is that there are points $x, y \in \mathbf{M}$ such that there is a twistor Z^α on L_x the dual planes of which contain all of $L_y^\perp$; this is in turn equivalent to $(x - y)^2 = 0$. Thus (4.4) is simply a restatement of the fact that the spacetime propagators are singular for $(x - y)^2 = 0$.

Another problem to which we can apply Theorem 4.1 is that of ϕ^4 scattering. To first order, this is a point interaction of four spin-0 zrm fields, and the amplitudes in spacetime are given by

$$(4.5) \quad \int \kappa(x) \chi(x) \psi(x) \theta(x) d^4x,$$

where $\theta, \chi \in Z_0(\overline{\mathbf{M}^+})$ are incoming and $\kappa, \psi \in Z_0(\overline{\mathbf{M}^-})$ are outgoing:

In twistor terms, we expect to find a map

$$(4.6) \quad H^1(\overline{\mathbf{P}_w^{*+}}; \mathcal{O}(-2)) \otimes H^1(\overline{\mathbf{P}_x^+}; \mathcal{O}(-2))$$

$$\otimes H^1(\overline{\mathbf{P}_y^{*+}}; \mathcal{O}(-2)) \otimes H^1(\overline{\mathbf{P}_z^+}; \mathcal{O}(-2)) \rightarrow \mathbb{C}.$$

Informally, this is given by [25]

$$(4.7) \quad (f, t, p, g) \rightarrow \oint \frac{f(W)t(X)p(Y)g(Z)}{(X \cdot W)(Z \cdot W)(X \cdot Y)(Z \cdot Y)} DW \wedge DX \wedge DY \wedge DZ.$$

In analogy with the inner product, we conclude that a twistor description of first order ϕ^4 scattering will follow from the construction of a “ ϕ^4 propagator”

$$(4.8) \quad \phi^4 \in H^4(\mathbf{P}_w^{*-} \times \mathbf{P}_x^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-2, -2, -2, -2)).$$

In light of (4.7), we expect ϕ^4 to satisfy

$$(4.9) \quad (X \cdot W)\phi^4 = (Z \cdot W)\phi^4 = (X \cdot Y)\phi^4 = (Z \cdot Y)\phi^4 = 0.$$

As in (3.16), it follows that we can find a

$$\sigma \in H^3(\Omega_{wx}^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-1, -1, -2, -2))$$

such that $\phi^4 = (X \cdot W)^{-1} \cdot \sigma$. Now

$$0 = (Z \cdot Y)\phi^4 = \frac{1}{X \cdot W} \cdot [(Z \cdot Y)\sigma],$$

but $(Z \cdot Y)\sigma \in H^3(\Omega_{wx}^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-1, -1, -1, -1))$, and since

$$H^3(\mathbf{P}_w^{*-} \times \mathbf{P}_x^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-1, -1, -1, -1)) = 0,$$

it follows that

$$\begin{aligned} & \cdot \frac{1}{X \cdot W} : H^3(\Omega_{wx}^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-1, -1, -1, -1)) \\ & \rightarrow H^4(\mathbf{P}_w^{*-} \times \mathbf{P}_x^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-; \mathcal{O}(-2, -2, -1, -1)) \end{aligned}$$

is an injection. It follows that $(Z \cdot Y)\sigma = 0$, so that $\sigma = (Z \cdot Y)^{-1} \cdot \tau$ for some

$$(4.10) \quad \tau \in H^2(\Omega_{wx}^- \times \Omega_{yz}^-; \mathcal{O}(-1, -1, -1, -1)).$$

We have presented these arguments in some detail because they cannot necessarily be extended. The process of reducing the cohomology degree by one while redefining the form on a subvariety is connected to integration over an S^1 in (4.7), and the *physically meaningful* contour for (4.7) is not an $S^1 \times S^1$ bundle over an $S^1 \times S^1$, but an $S^1 \times S^1$ bundle over an S^2 . This is described by Hodges [12], and summarized in [8, §V.1].

Instead, we apply Theorem 4.1. We first define the intermediate spaces

$$\begin{aligned} (4.11) \quad I &\equiv (\Omega_{wx}^- \times \Omega_{yz}^-) \cap \{Z \cdot W = 0\}, \\ J &\equiv (\Omega_{wx}^- \times \Omega_{yz}^-) \cap \{X \cdot Y = 0\}, \text{ and} \\ K &\equiv I \cap J. \end{aligned}$$

The comments of the preceding paragraph amount to the observation that τ is not of the form $(X \cdot Y)^{-1} \cdot (Z \cdot W)^{-1} \cdot \mu$ for any $\mu \in H^0(K; \mathcal{O})$. In fact, $H^0(K; \mathcal{O}) \simeq \mathbb{C}$ [8], so any $\mu \in H^0(K; \mathcal{O})$ extends to $\Omega_{wx}^- \times \Omega_{yz}^-$, and τ cannot be of this form.

We do, however, have the following

LEMMA 4.3. $(Z \cdot W)(X \cdot Y)$ is a section of the line bundle $\mathcal{O}(1, 1, 1, 1)$ over $\Omega_{wx}^- \times \Omega_{yz}^-$. If we set

$$(4.12) \quad \Sigma^- \equiv (\Omega_{wx}^- \times \Omega_{yz}^-) \cap \{(Z \cdot W)(X \cdot Y) = 0\},$$

then

$$(4.13) \quad H^2(\Omega_{wx}^- \times \Omega_{yz}^-, \Sigma^-; \mathbb{Z}) \simeq \mathbb{Z}.$$

PROOF. Since $H^1(\Omega^-; \mathbf{Z}) \simeq 0$ and Ω^- is connected,

$$H^2(\Omega^- \times \Omega^-; \mathbf{Z}) \simeq H^2(\Omega^-; \mathbf{Z}) \oplus H^2(\Omega^-; \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z}.$$

Now, Σ^- is the union of the intermediate spaces I and J , so we have the exact sequence

$$(4.14) \quad H^1(K; \mathbf{Z}) \rightarrow H^2(\Sigma^-; \mathbf{Z}) \rightarrow H^2(I; \mathbf{Z}) \oplus H^2(J; \mathbf{Z}) \rightarrow H^2(K; \mathbf{Z}) \rightarrow \cdots,$$

I is a fiber bundle, with base $\mathbf{P}_w^{*-} \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^- \cap \{Z \cdot Y = Z \cdot W = 0\}$ and fiber $\{X^\alpha \in \mathbf{P}^- \text{ such that } X \cdot W = 0\}$. The fiber is contractible, and the base is again a fiber bundle, with base $\mathbf{P}_w^{*-} \times \mathbf{P}_z^- \cap \{Z \cdot W = 0\} = \Omega_{wz}^-$ and (contractible) fiber $\{Y_\alpha \in \mathbf{P}^{*-} \text{ such that } Z \cdot Y = 0\}$, so $H^2(I; \mathbf{Z}) \simeq H^2(\Omega^-; \mathbf{Z}) \simeq \mathbf{Z}$. $H^2(J; \mathbf{Z}) \simeq \mathbf{Z}$ similarly.

K is also a fiber bundle, with base Ω_{wx}^- and fiber $\{(Z^\alpha, Y_\alpha) \in \Omega_{yz}^- \text{ such that } Z \cdot W = Y \cdot X = 0\}$. The fiber is contractible, so $H^1(K; \mathbf{Z}) = 0$ and $H^2(K; \mathbf{Z}) = \mathbf{Z}$. (4.14) now becomes

$$\begin{aligned} 0 &\rightarrow H^2(\Sigma^-; \mathbf{Z}) \rightarrow \mathbf{Z} \oplus \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \cdots, \\ (m, n) &\rightarrow m + n \end{aligned}$$

where we have identified the final map by considering the fibrations more carefully. It follows that $H^2(\Sigma^-; \mathbf{Z}) \simeq \mathbf{Z}$, and we can identify the restriction mapping from $\Omega_{wx}^- \times \Omega_{yz}^-$ to Σ^- as

$$\begin{aligned} H^2(\Omega_{wx}^- \times \Omega_{yz}^-; \mathbf{Z}) &\rightarrow H^2(\Sigma^-; \mathbf{Z}), \\ (m, n) &\rightarrow m + n. \end{aligned}$$

Since $H^1(\Sigma^-; \mathbf{Z}) \simeq 0$, the lemma follows. $\square$

The other assumptions of Theorem 4.1 are easily verified, and if we take k to be the generator of $H^2(\Omega_{wx}^- \times \Omega_{yz}^-, \Sigma^-; \mathbf{Z})$, then

$$(4.15) \quad P(k) \in H^2(\Omega_{wx}^- \times \Omega_{yz}^-; \mathfrak{O}(-1, -1, -1, -1))$$

is the desired element τ .

If we consider the line bundles $\mathfrak{O}(j, k, l, m)$ on $\mathbf{P}_w^{*-} \times \mathbf{P}_x^- \times \mathbf{P}_y^{*-} \times \mathbf{P}_z^-$, their Chern classes when restricted to $\Omega_{wx}^- \times \Omega_{yz}^-$ are $(j + k, l + m)$, and when restricted further to Σ^- are $j + k + l + m$. A possible choice for τ is therefore

$$\frac{1}{Z \cdot W X \cdot Y} \cdot \log(\mathfrak{O}(1, 0, 0, -1)).$$

It can be fairly argued at this point that although this construction leads to a map (4.6), we have no reason to believe that it corresponds to the ϕ^4 construction (4.5). We will deal with this point in §6, where we will show that

$$(4.16) \quad \phi^4 = \frac{1}{X \cdot W} \cdot \frac{1}{Z \cdot Y} \cdot \tau$$

does in fact correspond to the four-point field known to mediate ϕ^4 scattering (Hodges [13] and also [8]). We will also justify the appearance of Σ^- in Lemma 4.3 from a physical point of view (K may seem a more natural selection).

5. Nonprojective spaces. Since the bracket factors (3.28) are not defined for $k \leq 0$, the explicit expression (3.27) for the twistor propagators ϕ_n does not apply if $n < -1$. The construction of such an expression is our aim in this section.

An obvious choice is to define the bracket factor $(x)_0$ by

$$(x)_0 = \log x,$$

since this has the key property $d(x)_0/dx = (x)_1$. We would then have

$$(5.1) \quad \phi_{-2} = \log(Z \cdot W) \cdot \log(\Theta(1, -1)).$$

Unfortunately, there are problems with this definition. $\log(Z \cdot W)$ is not homogeneous in its arguments; a more serious difficulty arises because of the multi-valued nature of the logarithm.

We can deal with the homogeneity problem by pulling $\phi = \log(\Theta(1, -1)) \in H^1(\Omega^-; \Theta)$ back to the nonprojective space $\tilde{\Omega}^-$. We will abuse notation and write

$$(5.2) \quad \phi \in H^1(\tilde{\Omega}^-; \Theta)$$

as well; in fact, $\phi \in H^1(\tilde{\Omega}^-; \mathcal{H}(0, 0))$.

We now have

$$(5.3)$$

$$\phi_n = (Z \cdot W)_{n+2} \cdot \phi \in H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{H}(-n-2, -n-2)) \subset H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \Theta)$$

for $n \geq -1$. It is not hard to see from Theorem 2.4 that for $n \geq 1$,

$$(5.4)$$

$$H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{H}(-n-2, -n-2)) \simeq H^2(\mathbf{P}^- \times \mathbf{P}^{*-}; \Theta(-n-2, -n-2)),$$

and this enables us to recover the inner product pairing (3.9) from the elements (5.3). Alternatively, we can construct a pairing

$$(5.5) \quad H^1(\overline{\mathbf{T}}^+; \mathcal{H}(-n-2)) \otimes H^1(\overline{\mathbf{T}}^{*+}; \mathcal{H}(-n-2)) \rightarrow \mathbf{C}$$

directly by noting that for $(f, g) \in H^1(\overline{\mathbf{T}}^+; \mathcal{H}(-n-2)) \otimes H^1(\overline{\mathbf{T}}^{*+}; \mathcal{H}(-n-2))$, $f \cdot \phi_{-n} \cdot g \in H^6(\mathbf{T} \times \mathbf{T}^*; \mathcal{H}(-4, -4)) \subset H^6(\mathbf{T} \times \mathbf{T}^*; \Theta)$. In addition, there is a natural map

$$(5.6) \quad \int: H^3(\mathbf{T}; \Theta) \rightarrow \mathbf{C}$$

given by taking $f \in H^3(\mathbf{T}; \Theta)$ to

$$(5.7) \quad \int f \wedge \varepsilon$$

where $\varepsilon \in \Lambda^4 \mathbf{T}$ is as in §3 and we integrate the 7-form $f \wedge \varepsilon$ over an S^7 surrounding the origin in $\mathbf{C}^4$. The map (5.5) is thus given by

$$(f, g) \rightarrow \iiint f \cdot \phi_{-n} \cdot g.$$

This gives the same value as the construction in §3 by virtue of the following

LEMMA 5.1. *The following diagram commutes:*

$$\begin{array}{ccc} H^3(\mathbf{P}; \mathcal{O}(-4)) & \xrightarrow{\text{Serre}} & \mathbf{C} \\ \downarrow \pi^* & \nearrow & \\ H^3(\mathbf{T}; \mathcal{O}) & & \end{array}$$

In addition, $\int \pi^*: H^3(\mathbf{P}; \mathcal{O}(n)) \rightarrow \mathbf{C}$ is the zero map for $n \neq -4$.

PROOF. See Penrose [21]. $\square$

Since π^* and dot product clearly commute, it follows easily that the nonprojective construction gives the usual value and also that $0 \neq \phi \in H^1(\tilde{\Omega}^-; \mathcal{O})$.

For $n < -1$, we are now led to look for

$$(5.8) \quad \phi_n \in H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}),$$

which we require to satisfy the relations (3.15), and also

$$(5.9) \quad W_\alpha \frac{\partial \phi_n}{\partial W_\alpha} = Z^\alpha \frac{\partial \phi_n}{\partial Z^\alpha} = (-n-2)\phi_n.$$

In fact, for $n \leq -2$ we will have

$$(5.10) \quad \phi_n = \frac{(Z \cdot W)^{-n-2}}{(-n-2)!} \phi_{-2},$$

since, for $n \leq -3$, this implies

$$\begin{aligned} \frac{\partial \phi_n}{\partial Z^\alpha} &= W_\alpha \frac{(Z \cdot W)^{-n-3}}{(-n-1)!} \phi_{-2} + \frac{(Z \cdot W)^{-n-2}}{(-n-2)!} W_\alpha \left(\frac{1}{Z \cdot W} \cdot \phi \right) \\ &= W_\alpha \phi_{n+1} + \frac{W_\alpha}{(-n-2)!} (Z \cdot W)^{-n-3} \cdot \phi = W_\alpha \phi_{n+1}, \end{aligned}$$

because $(Z \cdot W)^{-n-3}$ is entire and the dot product annihilates entire functions.

If we define

$$(5.11) \quad \phi_+ \equiv \sum_{k=0}^{-\infty} \phi_k = \phi_0 + \phi_{-1} + e^{Z \cdot W} \phi_{-2} \in H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}),$$

we have the following

PROPOSITION 5.2. *Let $f \in H^1(\overline{\mathbf{P}}^+; \mathcal{O}(-m-2))$ and $g \in H^1(\overline{\mathbf{P}}^{*+}; \mathcal{O}(-n-2))$, where $m, n \geq 0$. Then*

$$(5.12) \quad \int \pi^* f \cdot \phi_+ \cdot \pi^* g = \begin{cases} 0 & \text{if } m \neq n, \\ \langle g | f \rangle & \text{if } m = n, \end{cases}$$

where $\langle g | f \rangle$ denotes the inner product of the states corresponding to f and g .

PROOF. This is an immediate consequence of the definition (5.11) and Lemma 5.1.

$\square$

In other words, ϕ_+ is a "universal" propagator for zrm fields of nonnegative helicity.

We are left with the problem of finding an explicit expression for ϕ_{-2} . (5.1) is still no good, since if we set

$$(5.13) \quad \tilde{B}^- \equiv T^- \times T^{*-} - \Omega^-,$$

then $Z \cdot W \in H^0(\tilde{B}^-; \mathcal{O}^*)$, and we have the exact segment

$$(5.14) \quad 0 \rightarrow H^0(\tilde{B}^-; \mathcal{O}) \rightarrow H^0(\tilde{B}^-; \mathcal{O}^*) \rightarrow Z = H^1(\tilde{B}^-; Z).$$

Unfortunately, $Z \cdot W$ has nonvanishing Chern class (its Chern class generates $H^1(\tilde{B}^-; Z)$), and $\log(Z \cdot W)$ therefore cannot be used as an element of $H^0(\tilde{B}^-; \mathcal{O})$.

Nonetheless, (5.1) does satisfy *formally* the key equations

$$\begin{aligned} \frac{\partial \phi_{-2}}{\partial Z^\alpha} &= \frac{W_\alpha}{(Z \cdot W)} \cdot \phi, & \frac{\partial \phi_{-2}}{\partial W_\alpha} &= \frac{Z^\alpha}{(Z \cdot W)} \cdot \phi, \quad \text{and} \\ Z^\alpha \frac{\partial \phi_{-2}}{\partial Z^\alpha} &= \frac{Z \cdot W}{Z \cdot W} \cdot \phi = 1 \cdot \phi = 0, & W_\alpha \frac{\partial \phi_{-2}}{\partial W_\alpha} &= 0, \end{aligned}$$

and we shall therefore continue to attempt to find a suitable interpretation for it.

Suppose that we were able to assign a precise meaning to the expression

$$(5.15) \quad (Z \cdot W)^\phi$$

as an element of $H^2(T^- \times T^{*-}; \mathcal{O}^*)$. Formally, we would then have

$$(5.16) \quad \log[(Z \cdot W)^\phi] = \log(Z \cdot W) \cdot \phi.$$

This hardly appears to constitute progress, since there are now sheeting problems in both (5.15) and (5.16). However, we can expect to deal with (5.16) by evaluating the Chern mapping $H^2(T^- \times T^{*-}; \mathcal{O}^*) \rightarrow H^3(T^- \times T^{*-}; Z)$; the key to the construction of (5.15) is the following

THEOREM 5.3. *If $i: Z \rightarrow \mathcal{O}$ is the natural injection, then there is a $k \in H^1(\tilde{\Omega}^-; Z)$ such that $\phi = ik \in H^1(\tilde{\Omega}^-; \mathcal{O})$.*

PROOF. Recall that transition functions for $e(\phi)$ are given by

$$f_{12} = \frac{A \cdot W}{B \cdot W} \frac{Z \cdot \bar{B}}{Z \cdot \bar{A}},$$

defined on $U_1 \cap U_2$, where $U_1 = \{A \cdot W \cdot Z \cdot \bar{B} \neq 0\}$ and $U_2 = \{B \cdot W \cdot Z \cdot \bar{A} \neq 0\}$ cover Ω^- . Since $A \cdot W \cdot Z \cdot \bar{B} \in H^0(\tilde{U}_1; \mathcal{O}^*)$, f_{12} is a coboundary nonprojectively, and $e(\phi) = 0 \in H^1(\tilde{\Omega}^-; \mathcal{O}^*)$.

It follows from the usual exact sequence

$$H^1(\tilde{\Omega}^-; Z) \xrightarrow{i} H^1(\tilde{\Omega}^-; \mathcal{O}) \xrightarrow{e} H^1(\tilde{\Omega}^-; \mathcal{O}^*)$$

that ϕ is in the image of i . $\square$

As in the projective case, T^- is a fiber bundle over E^4 , where the fiber is no longer a $\mathbf{CP}^1$ but is instead $\mathbf{C}^2 - \{0\}$. $\tilde{\Omega}^-$ is a fiber bundle over T^- , with fiber a deformation retract of $\mathbf{C}^*$. It follows that $H^p(T^-; Z) \simeq H^p(S^3; Z)$, and $H^1(\tilde{\Omega}^-; Z) \simeq Z$. Just as ϕ does not extend to either all of ambitwistor space or to $\mathbf{P}^- \times \mathbf{P}^{*-}$, so

we have

$$H^1(\tilde{\Omega}^-; \mathbf{Z}) \simeq 0 \simeq H^1(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbf{Z}).$$

($\tilde{\Omega}$ is a deformation retract of an S^5 bundle over an S^7 .)

For any complex manifold X , there is a natural map

$$(5.17) \quad \begin{aligned} H^0(X; \mathcal{O}^*) \otimes H^0(X; \mathbf{Z}) &\rightarrow H^0(X; \mathcal{O}^*), \\ (f, m) &\rightarrow f^m. \end{aligned}$$

The potential sheeting problems in (5.15) are a reflection of the fact that this does *not* extend to a map

$$\begin{aligned} H^0(X; \mathcal{O}^*) \otimes H^0(X; \mathcal{O}) &\rightarrow H^0(X; \mathcal{O}^*), \\ (f, g) &\rightarrow f^g, \end{aligned}$$

since f^g need only be defined on the universal covering space of X . In light of Theorem 5.3, however, we do not expect this difficulty to materialize.

The map (5.17) induces a cup product-type pairing

$$\cup : H^p(X; \mathcal{O}^*) \otimes H^q(X; \mathbf{Z}) \rightarrow H^{p+q}(X; \mathcal{O}^*).$$

As in the original construction of the dot product, we have

LEMMA 5.4. *Let X be a complex manifold, and U and V open in X . Then there is a natural pairing $H^p(U; \mathcal{O}^*) \otimes H^q(V; \mathbf{Z}) \rightarrow H^{p+q+1}(U \cup V; \mathcal{O}^*)$, denoted $(f, k) \rightarrow f^k$, such that if f or k extends to $U \cup V$, then $f^k = 0$. $\square$*

It follows that $(Z \cdot W)^\phi$ is indeed well defined as an element of $H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}^*)$. Unfortunately, we have the following

PROPOSITION 5.5. *The Chern class of $(Z \cdot W)^\phi$ is given by the dot product of the generators of $H^1(\tilde{B}^-; \mathbf{Z})$ and $H^1(\tilde{\Omega}^-; \mathbf{Z})$, and this dot product is nonzero.*

PROOF. We have the commutative diagram

$$\begin{array}{ccc} H^0(\tilde{B}^-; \mathcal{O}^*) & \xrightarrow{c} & H^1(\tilde{B}^-; \mathbf{Z}) \\ \downarrow f^\phi & & \downarrow \cdot \phi \\ H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}^*) & \xrightarrow{c} & H^3(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbf{Z}) \end{array}$$

where we have taken $\phi \in H^1(\tilde{\Omega}^-; \mathbf{Z})$. It follows that $c[(Z \cdot W)^\phi] = c(Z \cdot W) \cdot \phi$. However, it is easy to see that ϕ generates $H^1(\tilde{\Omega}^-; \mathbf{Z})$, and we have already remarked that $c(Z \cdot W)$ is a generator of $H^1(\tilde{B}^-; \mathbf{Z})$.

To show that this dot product is nonzero, we consider the following exact sequence due to Leray [16]:

(5.18)

$$\begin{array}{ccccccc} H^2(\tilde{B}^-; \mathbf{Z}) & \rightarrow & H^1(\tilde{\Omega}^-; \mathbf{Z}) & \xrightarrow{\cap^*} & H^3(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbf{Z}) & \xrightarrow{\rho} & H^3(\tilde{B}^-; \mathbf{Z}) \rightarrow H^2(\tilde{\Omega}^-; \mathbf{Z}) \\ 0 & \rightarrow & \mathbf{Z} & \rightarrow & \mathbf{Z} \oplus \mathbf{Z} & \rightarrow & \mathbf{Z} \rightarrow 0 \\ & & k & \rightarrow & (k, -k) & & \\ & & & & (m, n) & \rightarrow & m + n \end{array}$$

It is not hard to see that the map $\cap_*$ is simply dot product with a generator of $H^1(\tilde{B}^-; \mathbf{Z})$, and since $\cap_*$ is an injection, the proposition follows. In fact, using the coordinates of (5.18), the Chern class of $(Z \cdot W)^\phi$ is $(1, -1)$. $\square$

It is again tempting to discard (5.16) at this stage, but let us first see if it is possible to make precise the sense in which it is correct.

There is a sheaf map over $\mathbf{T}^- \times \mathbf{T}^{*-}$

$$d: \mathcal{O}^* \rightarrow \mathcal{O}_\alpha$$

given locally by

$$(5.19) \quad e^f \rightarrow \frac{1}{2\pi i} \frac{\partial f}{\partial Z^\alpha},$$

such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{Z} & \xrightarrow{i} & \mathcal{O} & \xrightarrow{e} & \mathcal{O}^* \rightarrow 0 \\ & & & & \downarrow \partial/\partial Z^\alpha & & \downarrow d \\ & & & & \mathcal{O}_\alpha & & \end{array}$$

Similarly, we define $d^*: \mathcal{O}^* \rightarrow \mathcal{O}^\alpha$ by $e^f \rightarrow 1/2\pi i \partial f / \partial W_\alpha$.

d induces a map $d: H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}^*) \rightarrow H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}_\alpha)$, and

$$(5.20) \quad \begin{aligned} d[(W \cdot Z)^\phi] &= \frac{W_\alpha}{Z \cdot W} \cdot \phi = W_\alpha \phi_{-1} = \frac{\partial \phi_{-2}}{\partial Z^\alpha} = d[e(\phi_{-2})], \\ d^*[(W \cdot Z)^\phi] &= \frac{Z^\alpha}{Z \cdot W} \cdot \phi = Z^\alpha \phi_{-1} = \frac{\partial \phi_{-2}}{\partial W_\alpha} = d^*[e(\phi_{-2})]. \end{aligned}$$

We have shown the following

PROPOSITION 5.6. *Let $\mu = e(\phi_{-2})/(Z \cdot W)^\phi \in H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathcal{O}^*)$. Then*

$$(5.21) \quad d\mu = d^*\mu = 0,$$

and

$$(5.22) \quad c(\mu) = (-1, 1).$$

Further, for any μ satisfying (5.21) and (5.22), ϕ_{-2} is the pullback along e of $\mu(Z \cdot W)^\phi$. (This pullback is unique, since $H^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbf{Z}) = 0$.)

PROOF. (5.21) follows from (5.20), and we also have

$$c(\mu) = ce(\phi_{-2}) - ce(Z \cdot W)^\phi = (-1, 1).$$

Conversely, given μ satisfying (5.21) and (5.22), it is not hard to see that $(1/2\pi i) \log[\mu(Z \cdot W)^\phi]$ satisfies (3.15) and, by Theorem 3.6, must therefore be equal to ϕ_{-2} . $\square$

In practice, there will generally be derivatives involved in the construction of the spacetime fields associated to any positive homogeneity propagator, and an explicit representation of μ is not needed. We will, however, give a purely twistorial construction of it.

THEOREM 5.7. *Let $p: X \rightarrow B$ be a fibration of a complex manifold X , where B is n -dimensional and noncompact. If we fix $k \in H^{n+1}(B; \mathbf{Z})$, then there exists an element $\nu \in H^n(X; \mathcal{O}^*)$ such that $d\nu = 0$ and $c(\nu) = p^*k \in H^{n+1}(X; \mathbf{Z})$, where d is as in (5.19).*

PROOF. We have the exact segment

$$(5.23) \quad H^n(B; \mathcal{O}) \rightarrow H^n(B; \mathcal{O}^*) \rightarrow H^{n+1}(B; \mathbf{Z}) \rightarrow H^{n+1}(B; \mathcal{O}).$$

The last group vanishes because B is only n -dimensional, and the first group also vanishes, by virtue of a theorem due to Siu [27] which states that $H^n(B; \mathcal{S}) = 0$ if B is n -dimensional and noncompact and $\mathcal{S}$ is coherent analytic. (5.23) therefore becomes

$$c: H^n(B; \mathcal{O}^*) \simeq H^{n+1}(B; \mathbf{Z}),$$

and we can pick $\nu_0 \in H^n(B; \mathcal{O}^*)$ such that $c(\nu_0) = k$. Since $H^n(B; \mathcal{O}) = 0$, $d\nu_0 = 0$ (d being defined as in (5.19)).

We now set $\nu = p^*(\nu_0)$. Since the diagram

$$\begin{array}{ccc} H^n(B; \mathcal{O}^*) & \rightarrow & H^{n+1}(B; \mathbf{Z}) \\ \downarrow p^* & & \downarrow p^* \\ H^n(X; \mathcal{O}^*) & \rightarrow & H^{n+1}(X; \mathbf{Z}) \end{array}$$

commutes, $c(\nu) = p^*k$. In addition, if we extend the coordinates on B to a set of coordinates on X , it is clear that p^* and d commute in a suitable sense, so that $d\nu = p^*d\nu_0 = 0$. $\square$

We will refer to ν as a $c(\nu)$ -normalizer for X . If $c(\nu)$ generates $H^{n+1}(X; \mathbf{Z})$, we will simply call ν an n -normalizer for X .

COROLLARY 5.8. *Let L be a line in $\mathbf{P}$. Then $\mathbf{T} - \tilde{L}$ has a 2-normalizer.*

PROOF. $\mathbf{P} - L$ is a $\mathbf{C}^2$ bundle over a $\mathbf{CP}^1$, where the base is the set of planes through L and the fiber is the $\mathbf{C}^2$ obtained by removing the line L from such a plane. Nonprojectively, $\mathbf{T} - \tilde{L}$ is a $\mathbf{C}^2$ bundle over $\mathbf{C}^2 - \{0\}$, and we therefore see that $H^3(\mathbf{T} - \tilde{L}; \mathbf{Z}) \simeq H^3(\mathbf{C}^2 - \{0\}; \mathbf{Z}) \simeq \mathbf{Z}$. $\square$

The projection from $\mathbf{T} - \tilde{L}$ to the base is given explicitly by

$$\begin{aligned} p: \mathbf{T} - \tilde{L} &\rightarrow \tilde{L}^\perp, \\ z^\alpha &\rightarrow \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha A^\beta B^\gamma, \end{aligned}$$

where $L^\perp$ is the line in $\mathbf{P}^*$ dual to L and A and B are arbitrary distinct twistors on L .

PROPOSITION 5.9. *Let ϕ be a generator of $H^1(\tilde{\Omega}^-; \mathbf{Z})$ and ν and ν^* 2-normalizers for $\mathbf{T}^-$ and $\mathbf{T}^{*-}$ respectively. The twistor propagators are then given by*

$$(5.24a) \quad \phi_n = (Z \cdot W)_{n+2} \cdot \phi \quad \text{for } n \geq -1,$$

$$(5.24b) \quad \phi_n = \frac{(Z \cdot W)^{-n-2}}{(-n-2)!} \log \left[\frac{\nu^*}{\nu} (Z \cdot W)^\phi \right] \quad \text{for } n \leq -2.$$

PROOF. We have already established (5.24a). For (5.24b), we have

$$\begin{aligned} c\left(\frac{\nu^*}{\nu}\right) &= c(\nu^*) - c(\nu) = (-1, 1), \\ d\left(\frac{\nu^*}{\nu}\right) &= -d(\nu) = 0 = d^*\left(\frac{\nu^*}{\nu}\right). \end{aligned}$$

Now apply Proposition 5.6. $\square$

The notation in (5.24b) obscures the fact that ϕ_{-2} obeys the fundamental equations (3.15). Another way to think of this construction is as follows: Suppose $\mathcal{U} = \{U_i\}$ covers $\tilde{B}^- = \{Z \cdot W \neq 0\} \cap \mathbf{T}^- \times \mathbf{T}^{*-}$ in such a way that $\log(Z \cdot W)$ is well behaved on each U_i . Now $\log(Z \cdot W) \in C^0(\tilde{B}^-; \mathbb{C})$ is a 0-cochain on $\tilde{B}^-$ which does not, of course, satisfy the cocycle condition, since $\log(Z \cdot W)$ is not globally defined on $\tilde{B}^-$. Similarly, $\log(Z \cdot W) \cdot \phi \in C^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbb{C})$ also does not satisfy the cocycle condition, as $\log(\nu^*/\nu) \in C^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbb{C})$ does not. However, we can choose the logarithms so that $\log(Z \cdot W) \cdot \phi + \log(\nu^*/\nu)$ *does* satisfy the cocycle condition, and $\partial \log(\nu^*/\nu)/\partial Z^\alpha \in C^2(\mathbf{T}^- \times \mathbf{T}^{*-}; \mathbb{C}_\alpha)$ is actually a coboundary, as is $\partial \log(\nu^*/\nu)/\partial W_\alpha$. The first of these statements follows from the fact that $c[\nu^*(Z \cdot W)^*/\nu] = 0$, while the second is a consequence of $d(\nu^*/\nu) = 0 = d^*(\nu^*/\nu)$. We can therefore rewrite (5.24b) as

$$(5.25) \quad \phi_n = \frac{(Z \cdot W)^{-n-2}}{(-n-2)!} \left[\log(Z \cdot W) \cdot \phi + \log \frac{\nu^*}{\nu} \right] \quad \text{for } n \leq -2.$$

PROPOSITION 5.10. *The universal propagator of Proposition 5.2 is given explicitly by*

$$\phi_+ = \left[\frac{1}{Z \cdot W} - \frac{1}{(Z \cdot W)^2} \right] \cdot \phi + e^{Z \cdot W} \left[\log(Z \cdot W) \cdot \phi + \log \frac{\nu^*}{\nu} \right]. \quad \square$$

6. Spacetime expressions. It is possible to evaluate simple scattering amplitudes directly using methods such as those of the last two sections, but to see that we are in fact performing the usual spacetime calculations, we need results such as the following

PROPOSITION 6.1. *The twistor propagator $\phi_0(W_\alpha, Z^\alpha)$ corresponds to the spacetime field $1/(w-z)^2$, up to proportionality.*

PROOF. The field associated to ϕ_0 is given by [15]

$$(6.1) \quad \frac{1}{(2\pi i)^3} \oint_{\rho_{w,z}} \frac{\Delta W \wedge \Delta Z}{(Z \cdot W)^2} \log \left(\frac{A \cdot W}{B \cdot W} \frac{Z \cdot \bar{B}}{Z \cdot \bar{A}} \right).$$

If we choose coordinates η_A for L_w and $\nu_{A'}$ for L_z , and set $\mu^{A'} = (w-z)^{A'} \eta_{A'}$, this becomes

$$(6.2) \quad \frac{1}{(2\pi i)^3} \frac{1}{(w-z)^2} \oint \frac{\Delta \mu \wedge \Delta \nu}{(\mu \cdot \nu)^2} \log \left(\frac{\mu \cdot \alpha}{\mu \cdot \beta} \frac{\tilde{\beta} \cdot \nu}{\tilde{\alpha} \cdot \nu} \right),$$

where $\mu \cdot \nu = \mu^{A'} \nu_{A'} = -\mu_{A'} \nu^{A'}$, $\Delta \mu = \mu_{A'} d\mu^{A'}$, etc. If we now write

$$\mu_{A'} = \begin{pmatrix} 1 \\ u \end{pmatrix}, \quad \nu_{A'} = \begin{pmatrix} 1 \\ v \end{pmatrix},$$

(6.2) becomes

$$\frac{1}{(2\pi i)^3} \frac{1}{(w-z)^2} \oint \frac{du dv}{(u-v)^2} \log \left(\frac{a+u}{b+u} \frac{\tilde{b}+v}{\tilde{a}+v} \right).$$

Integrating by parts gives

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \frac{1}{(w-z)^2} \oint \frac{du dv}{u-v} \left(\frac{1}{a+u} - \frac{1}{b+u} \right) \\ &= \frac{1}{(2\pi i)^2} \frac{1}{(w-z)^2} \oint du \left(\frac{1}{a+u} - \frac{1}{b+u} \right) = \frac{1}{\pi i (w-z)^2}. \end{aligned}$$

The final integration is over a contour which separates the two poles because the original interpretation of the logarithm in (6.1) was as an H^1 . $\square$

Of course, $1/(w-z)^2$ is none other than the usual propagator for massless scalar fields. Similarly, we have

PROPOSITION 6.2. *Let $w, y \in \mathbf{M}^+$ and $x, z \in \mathbf{M}^-$. Now set*

$$\begin{aligned} (6.3) \quad & a^{AA'} = (x-w)^{AA'}, \\ & b^{AA'} = (z-w)^{AA'}, \\ & c^{AA'} = (x-y)^{AA'}, \\ & d^{AA'} = (z-y)^{AA'}, \end{aligned}$$

and

$$\lambda^{B'}_{A'} = a^{AB'} b_{AC'} c_{BA'} d^{BC'}.$$

If $t = \lambda^{A'}_{A'}$ is the trace of λ , and $\Delta = [\lambda^{A'B'}(\lambda_{A'B'} + \lambda_{B'A'})]^{1/2}$ is the discriminant of its characteristic equation, then the 4-point field associated to ϕ^4 is (up to proportionality)

$$(6.4) \quad \frac{\log((t-\Delta)/(t+\Delta))}{2\pi i \Delta}.$$

PROOF. We must evaluate

$$(6.5) \quad \frac{1}{(2\pi i)^4} \oint \rho_{wxyz} \frac{\Delta W \wedge \Delta X \wedge \Delta Y \wedge \Delta Z}{(X \cdot W)(Z \cdot Y)(Z \cdot W)(X \cdot Y)} \log(\Theta(1, 0, 0, -1)).$$

As in (6.2), this becomes

$$\begin{aligned} (6.6) \quad & \frac{1}{(2\pi i)^4} \oint \frac{\Delta \zeta \wedge \Delta \pi \wedge \Delta \xi \wedge \Delta \mu}{(a^{AA'} \pi_A \zeta_A)(b^{AA'} \mu_A \zeta_A)(c^{AA'} \pi_A \xi_A)(d^{AA'} \mu_A \xi_A)} \log(\Theta(1, 0, 0, -1)) \\ &= \frac{1}{(2\pi i)^4} \oint \frac{\Delta \nu \wedge \Delta \pi \wedge \Delta \rho \wedge \Delta \sigma}{(\pi \cdot \nu)(\pi \cdot \rho)(\sigma \cdot \rho)(\sigma \cdot \lambda \cdot \nu)} \log(\Theta(1, 0, 0, -1)) \\ &= \frac{1}{(2\pi i)^2} \oint \frac{\Delta \pi \wedge \Delta \sigma}{(\pi \cdot \sigma)(\pi \cdot \lambda \cdot \sigma)} \log(\Theta(1, -1)), \end{aligned}$$

where we have changed variables a few times and integrated over two simple poles.

We now assume without loss of generality that

$$\pi_{A'} = \begin{pmatrix} 1 \\ p \end{pmatrix}, \quad \sigma_{A'} = \begin{pmatrix} 1 \\ s \end{pmatrix} \quad \text{and} \quad \lambda^{A'B'} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.$$

(6.6) becomes

$$\begin{aligned} & \frac{1}{(2\pi i)^3} \oint \frac{dp \, ds}{(p-s)(fp-es)} \log \left(\frac{a+p}{b+p} \frac{\tilde{b}+s}{\tilde{a}+s} \right) \\ &= \frac{1}{(2\pi i)^3} \oint \frac{dp \, ds}{s(f-e)} \left(\frac{1}{p-s} - \frac{f}{fp-es} \right) \log \left(\frac{a+p}{b+p} \frac{\tilde{b}+s}{\tilde{a}+s} \right) \\ &= \frac{1}{(2\pi i)^2} \oint \frac{ds}{s(f-e)} \left[\log \left(\frac{a+s}{b+s} \frac{\tilde{b}+s}{\tilde{a}+s} \right) - \log \left(\frac{es+fa}{es+fb} \frac{s+\tilde{b}}{s+\tilde{a}} \right) \right] \\ &= \frac{1}{(2\pi i)^2} \oint \frac{ds}{s(f-e)} \log \left(\frac{a+s}{b+s} \frac{es+fb}{es+fa} \right). \end{aligned}$$

Examination of the original logarithm shows that the branch cuts in the above expression connect $a+s=0$ to $es+fa=0$ and connect $b+s=0$ to $es+fb=0$; the contour surrounds each of these cuts. Since the logarithm jumps by $2\pi i$ from one side of the cut to the other, the field is given by

$$\frac{1}{2\pi i} \left[\int_{-fa/e}^{-a} + \int_{-fb/e}^{-b} \right] \frac{ds}{s(f-e)} = \frac{1}{\pi i} \frac{\log(e/f)}{f-e} = \frac{1}{\pi i \Delta} \log \left(\frac{t-\Delta}{t+\Delta} \right). \quad \square$$

It has been shown by Hodges [12, 13] that this field does, in fact, mediate ϕ^4 scattering.

It is also possible to give direct twistorial proofs of well-known spacetime results. For example, we have

PROPOSITION 6.3. *The twistor propagators $\phi_n \in H^2(\mathbf{P}^- \times \mathbf{P}^{*-}; \mathfrak{O}(-n-2, -n-2))$ correspond to fields $\phi_n(x, z)$ satisfying*

$$(6.7) \quad \phi_n(x, z) = (-1)^n \phi_n(z, x).$$

PROOF. We need the following lemma.

LEMMA 6.4. *Let Δ be the diagonal in $\mathbf{P} \times \mathbf{P}$, and suppose that*

$$\lambda(X, Z) \in H^2(\mathbf{P} \times \mathbf{P} - \Delta; \mathfrak{O}(-2, -2))$$

satisfies $\lambda(X, Z) = \mp \lambda(Z, X)$. Then λ corresponds to a two-point field $\kappa(x, z)$ with $\kappa(x, z) = \pm \kappa(z, x)$.

PROOF.

$$\begin{aligned} \kappa(x, z) &= \oint \rho_{xz} \lambda(X, Z) \Delta X \wedge \Delta Z \\ &= \mp \oint \rho_{zx} \lambda(Z, X) \Delta X \wedge \Delta Z \\ &= \pm \oint \rho_{zx} \lambda(Z, X) \Delta Z \wedge \Delta X = \pm \kappa(z, x). \quad \square \end{aligned}$$

PROOF OF PROPOSITION 6.3. We set

$$\begin{aligned} W_\alpha &= (\eta_A, \xi^{A'}), \\ Z^\alpha &= (\omega^A, \pi_{A'}), \\ X^\alpha &= (\nu^A, \mu_{A'}). \end{aligned}$$

It is easy to show that $\phi_n(w, z)$ corresponds to the same zrm field as does

$$\begin{aligned} (6.8) \quad & (-1)^n \pi_{A'} \cdots \pi_{B'} \eta_A \cdots \eta_B \phi_n(W, Z) \\ &= (-1)^n \pi_{A'} \cdots \pi_{B'} \frac{\partial}{\partial \omega^A} \cdots \frac{\partial}{\partial \omega^B} \phi_0(W, Z) \\ &\in H^2(\mathbf{P}^- \times \overline{\mathbf{P}^{*-}}; \mathcal{O}_{A' \cdots B' A \cdots B}(-2, -2)) \end{aligned}$$

where we have extended ϕ_n to $\mathbf{P}^- \times \overline{\mathbf{P}^{*-}}$. (This is possible since, for example, $H^1(\mathbf{T}^- \times \overline{\mathbf{T}^{*-}} \cap \{Z \cdot W = 0\}; \mathbf{Z}) \simeq \mathbf{Z}$.)

If $\phi'_0 \in H^2(\mathbf{P}^+ \times \overline{\mathbf{P}^{*+}}; \mathcal{O}(-2, -2))$ is a positive-frequency propagator, we can consider

$$\begin{aligned} \phi_0(W, Z) \cdot \phi'_0(W, X) &\in H^5(\mathbf{P}_x^+ \times \mathbf{P}_w^* \times \mathbf{P}_z^-; \mathcal{O}(-2, -4, -2)) \\ &\simeq H^2(\mathbf{P}_x^+ \times \mathbf{P}_z^-; \mathcal{O}(-2, -2)) \quad (\text{Serre duality}). \end{aligned}$$

We thus set

$$(6.9) \quad \psi_0(X, Z) = \phi_0(W, Z) \cdot \phi'_0(W, X);$$

$\psi_0(X, Z) \in H^2(\mathbf{P}_x^+ \times \mathbf{P}_z^-; \mathcal{O}(-2, -2))$ corresponds to the same zrm field as ϕ_0 does, and ψ_0 extends to $\mathbf{P} \times \mathbf{P} - \Delta$. The extension to $\mathbf{P}_x^- \times \mathbf{P}_z^+$ is given by

$$(6.10) \quad \psi_0(X, Z) = \phi'_0(W, Z) \cdot \phi_0(W, X).$$

Combining this result with (6.8) and Lemma 6.4, we see that we want to show

$$\begin{aligned} (6.11) \quad & (-1)^n \pi_{A'} \cdots \pi_{B'} \frac{\partial}{\partial \omega^A} \cdots \frac{\partial}{\partial \omega^B} \psi_0(X, Z) \\ &= (-1)^n (-1)^{n+1} \mu_{A'} \cdots \mu_{B'} \frac{\partial}{\partial \nu^A} \cdots \frac{\partial}{\partial \nu^B} \psi_0(Z, X). \end{aligned}$$

The left-hand side of this equation is

$$\begin{aligned} & (-1)^n \left[\pi_{A'} \cdots \pi_{B'} \frac{\partial}{\partial \omega^A} \cdots \frac{\partial}{\partial \omega^B} \phi_0(W, Z) \right] \cdot \phi'_0(W, X) \\ &= (-1)^n \left[\frac{\partial}{\partial \xi^{A'}} \cdots \frac{\partial}{\partial \xi^{B'}} \eta_A \cdots \eta_B \phi_0(W, Z) \right] \cdot \phi'_0(W, X). \end{aligned}$$

Since

$$\frac{\partial}{\partial \xi^{A'}} [\phi_0(W, Z) \cdot \phi'_0(W, X)] = 0,$$

we can “integrate by parts” n times to get

$$\begin{aligned} & (-1)^n (-1)^n \phi_0(W, Z) \cdot \left[\frac{\partial}{\partial \xi^{A'}} \cdots \frac{\partial}{\partial \xi^{B'}} \eta_A \cdots \eta_B \right] \phi'_0(W, X) \\ &= \phi_0(W, Z) \cdot \left[\mu_{A'} \cdots \mu_{B'} \frac{\partial}{\partial \nu^A} \cdots \frac{\partial}{\partial \nu^B} \phi'_0(W, X) \right] \\ &= \mu_{A'} \cdots \mu_{B'} \frac{\partial}{\partial \nu^A} \cdots \frac{\partial}{\partial \nu^B} \psi_0(X, Z). \end{aligned}$$

(6.11) therefore reduces to $\psi_0(Z, X) = -\psi_0(X, Z)$, or, by (6.10),

$$\phi_0(W, Z) \cdot \phi'_0(W, X) = -\phi'_0(W, X) \cdot \phi_0(W, Z).$$

This follows immediately from the fact that H^2 's anticommute under dot product. $\square$

7. Physical interpretation. The primary purpose of this paper has been to show that certain geometrical objects defined on products of twistor spaces can be used to describe interactions of massless particles on Minkowski space. The specific examples we have given have dealt with two such objects,

$$\phi \in H^1(\tilde{\Omega}^-; \mathbf{Z}),$$

used in the description of the twistor propagators, and

$$\omega \in H^1(\tilde{\Omega}_{wx}^- \times \tilde{\Omega}_{yz}^- \cap \{(Z \cdot W)(X \cdot Y) = 0\}; \mathbf{Z}),$$

which can be used in an analogous fashion to describe ϕ^4 scattering. We conclude by describing a possible direct interpretation of these objects on spacetime.

Consider, for example, the twistor propagator

$$(7.1) \quad \phi_{-1} = \frac{1}{Z \cdot W} \cdot \phi$$

used in the construction

$$\begin{aligned} & H^1(\overline{\mathbf{P}^+}; \mathfrak{O}(-3)) \otimes H^1(\overline{\mathbf{P}^{*+}}; \mathfrak{O}(-3)) \rightarrow \mathbf{C}, \\ & (f, g) \rightarrow \text{Serre}(f \cdot \phi_{-1} \cdot g). \end{aligned}$$

The appearance of the factor $1/Z \cdot W$ (as in (7.1)) in this construction dates back to Penrose [25]. By using Cauchy's theorem, he shows that this factor corresponds to the fact that the fields f and g are interacting *at a point*. Indeed, the Feynman diagram for this process is:

$$\begin{array}{c} \uparrow g(W_\alpha) \\ \bullet \\ \uparrow f(Z^\alpha) \end{array}$$

We therefore assign a physical interpretation to (7.1) as follows:

(a) The fact that ϕ is defined on $\{Z \cdot W = 0\}$ indicates that the Z^α particle and the W_α particle are interacting.

(b) The inclusion of the factor $1/Z \cdot W$ indicates that the interaction is pointlike. The ϕ^4 propagator

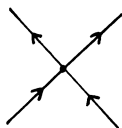
$$(7.2) \quad \phi^4 = \frac{1}{X \cdot W} \cdot \frac{1}{Z \cdot Y} \cdot \frac{1}{Z \cdot W X \cdot Y} \cdot \omega$$

can be analyzed similarly. Since $Z \cdot W X \cdot Y = 0$ if and only if $Z \cdot W = 0$ or $X \cdot Y = 0$, we have:

(a) The X^α and W_α particles are interacting, as are the Z^α and Y_α particles. Furthermore, *either* the Z^α and W_α particles *or* the X^α and Y_α particles are also interacting.

(b) All of the interactions are pointlike.

In fact, this is just enough information for us to conclude that all four particles are interacting at a point, and that the associated Feynman diagram is:



If we were to replace (a) with

(a') All pairs of particles corresponding to a twistor and a dual twistor are interacting,

we would in some sense have “overdetermined” the system. It is for this reason that Σ^- (and not K) appears in Lemma 4.3. (See also the comments in the paragraph preceding Lemma 4.3.)

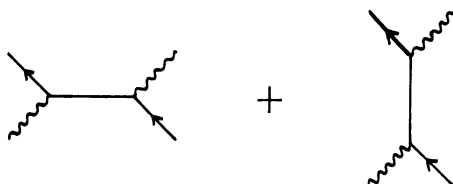
There are a variety of directions in which one could proceed at this point. More involved first-order processes should be considered, such as Møller scattering, which is discussed in [21]. The creation-annihilation channel



should be tractable, while the exchange channels



which suffer from an infrared divergence, may not be. In Compton scattering (treated by Hodges [14]), on the other hand, it is only the *sum*



which is physically meaningful, rather than the individual diagrams such as (7.3) and (7.4). It will be interesting to understand this geometrically.

Finally, higher-order processes will need to be dealt with. Since constructions such as those we have described necessarily give well-defined maps into $\mathbb{C}$, twistor descriptions of these processes should be free of the divergences which appear in quantum field theory.

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